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Tikhonov-Regularization of Ill-Posed Linear Operator Equations on Closed Convex Sets

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In this paper we derive conditions under which the constrained Tikhonov-regularized solutions $x_{\alpha,C}$ of an ill-posed linear operator equation $Tx=y$ (i.e., $x_{\alpha,C}$ is the minimizing element of the functional $\|Tx-y\|^2+\alpha\|x\|^2$ in the closed convex set C) converge to the best-approximate solution of the equation in C with rates $o(\alpha^{1/2})$ and $O(\alpha)$, respectively.

1. INTRODUCTION

In many problems arising in practice one has to solve linear operator equations

$$Tx=y,$$

where x and y are elements of real Hilbert spaces X and Y, respectively, and T is a linear bounded operator from X into Y. By a solution of the equation $Tx=y$ we always mean the best-approximate solution $T^\dagger y$, where $T^\dagger$ is the Moore–Penrose inverse of T. Unfortunately, in general $T^\dagger y$ does not depend continuously on the right-hand side y. A prominent example for the equation $Tx=y$ is a Fredholm integral equation of the first kind,

$$\int_0^1 k(t,s)\,x(s)\,ds=y(t), \qquad t\in[0,1],$$

$x, y\in L^2[0,1]$, $k\in L^2([0,1]^2)$. Here $T^\dagger$ is bounded if and only if k is a degenerate kernel. Therefore, one has to regularize the equation $Tx=y$. A well-known and effective regularization method is Tikhonov-regularization, where the functional $\|Tx-y\|^2+\alpha\|x\|^2$, $\alpha>0$, is minimized in X (cf., e.g., [4]).

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Often one knows a priori that the solution $T^\dagger y$ is an element of a certain subset of X, e.g., it is clear that density functions will never assume negative values. On the other hand, one is often not interested in the solution $T^\dagger y$, but in the best-approximate solution on a certain set C, which we assume to be closed and convex in the following. In this situation it is reasonable to require that the regularized solutions should have the same properties as the unknown exact solution, e.g., it should be an element of C. Hence, we regularize the problem

$$Tx = y \wedge x \in C$$

by minimizing the Tikhonov-functional $\|Tx - y\|^2 + \alpha \|x\|^2$, $\alpha > 0$, on C. We call the solution $x_{\alpha,C}$ of this minimum problem "constrained Tikhonov-regularized solution."

In Section 2 we deal with convergence and stability of constrained Tikhonov-regularized solutions. Similar results about convergence and stability of constrained Tikhonov-regularized solutions have been developed in [5, 6] in a somewhat different way as presented here. In Section 3 we summarize well-known convergence order results for the unconstrained case (cf., e.g., [2–4, 9]). In Section 4 we show that the condition "$T^\dagger y \in R(T^*)$," which is sufficient for the convergence rate $o(\alpha^{1/2})$ in the unconstrained case, can be replaced by "$x_{0,C} \in R(P_C T^*)$" in the constrained case, where P_C is the metric projector onto C and $x_{0,C}$ is the best-approximate solution of $Tx = y$ on C. The main theorem of Section 4 is Theorem 4.2. It is much more difficult to find an analogous condition to "$T^\dagger y \in R(T^*T)$," which implies the convergence rate $O(\alpha)$ in the constrained case, too. The condition "$x_{0,C} \in R(P_C T^*T)$" is only necessary but not sufficient for the convergence rate $O(\alpha)$.

Only if we require further conditions on the set C and $x_{0,C}$, we can guarantee the convergence rate $O(\alpha)$ in the constrained case (see Theorem 5.13). If, for example, C has a twice continuously Fréchet-differentiable boundary in a neighbourhood of $x_{0,C} \in \partial C$, it is sufficient for the convergence rate $O(\alpha)$ that the second derivative of the boundary in $x_{0,C}$ is positive definite and that $\tilde{P} x_{0,C} \in R(\tilde{P} T^* T \tilde{P})$, where $\tilde{P}$ is the orthogonal projector onto the hyperplane through the origin, which is parallel to the tangential plane to ∂C in $x_{0,C}$. Since the proofs of the results in Section 5 concerning with the convergence rate $O(\alpha)$ are very technical, they will be omitted here. For the proofs of the results in Section 5 see [7].

2. Constrained Tikhonov-Regularization

Throughout this paper let X and Y be real Hilbert spaces, $T: X \to Y$ a bounded linear operator; the set of all bounded linear operators on X into

Y will be denoted by $L(X, Y)$. The inner products and norms in X and Y, though in general different, will both be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. We consider the problem of solving

$$Tx = y \wedge x \in C \tag{2.1}$$

with $y \in Y$ and $\emptyset \neq C (\subset X)$ a convex closed set. We define now what we mean by "the solution" of (2.1).

DEFINITION 2.1. $x_{0,C} \in C$ is called "*C-best-approximate solution*" of (2.1) if

$$\|Tx_{0,C} - y\| = \inf\{\|Tx - y\| / x \in C\}$$

and

$$\|x_{0,C}\| = \inf\{\|x\| / x \in C \wedge \|Tx - y\| = \|Tx_{0,C} - y\|\}.$$

Thus, a C-best-approximate solution minimizes the norm of the residual on C and has minimal norm among all minimizers. In the following proposition we show that the C-best-approximate solution of (2.1) only exists for certain elements $y \in Y$.

PROPOSITION 2.2. *Let R be the metric projector of Y onto $\overline{T(C)}$. Then a C-best-approximate solution exists if and only if $Ry \in T(C)$; it is then unique.*

Proof. Obviously $\hat{x} \in C$ minimizes $\|Tx - y\|$ on C if and only if $\hat{x} \in C$ minimizes $\|Tx - y\|^2$ on C. Since C is closed and convex and since $\|Tx - y\|^2$ is a convex and Fréchet-differentiable functional, by the Kuhn–Tucker theory this is equivalent to

$$\hat{x} \in C \qquad \text{and} \qquad (T\hat{x} - y, u - T\hat{x}) \geqslant 0 \qquad \text{for all } u \in \overline{T(C)}. \tag{2.2}$$

Since $\overline{T(C)}$ is closed and convex and since $g(u) := \|u - y\|^2$ is a strictly convex and Fréchet-differentiable functional with $\nabla g(u) = 2(u - y)$ and $\lim_{\|u\| \to \infty} g(u) = \infty$, Ry is defined as the unique element in $\overline{T(C)}$, for which

$$(Ry - y, u - Ry) \geqslant 0 \qquad \text{for all } u \in \overline{T(C)} \tag{2.3}$$

holds. Now it follows with (2.2) and (2.3) that

$$\hat{x} \in C \text{ minimizes } \|Tx - y\| \text{ on } C \quad \Leftrightarrow \quad \hat{x} \in C \wedge T\hat{x} = Ry. \tag{2.4}$$

Let $K := \{\hat{x} \in C / \hat{x} \text{ minimizes } \|Tx - y\| \text{ on } C\}$. Then (2.4) implies that $K \neq \emptyset$ if and only if $Ry \in T(C)$. Since $\|Tx - y\|$ is a convex functional on the convex closed set C, K is closed and convex. Therefore, if $K \neq \emptyset$, there exists a unique element of minimal norm in K. Together with Definition 2.1 we have: $Ry \in T(C)$ is equivalent to the existence of a unique C-best-approximate solution. ∎

If $C = X$, the condition $Ry \in T(C)$ is equivalent to $y \in D(T^\dagger) = R(T) + R(T)^\perp$ and the (X-) best-approximate solution is then given by $T^\dagger y$, where $T^\dagger$ is the Moore–Penrose inverse of T. $T^\dagger$ is continuous if and only if $R(T)$ is closed (cf., e.g., [4]). Therefore, the problem of determining the best-approximate solution $T^\dagger y$ is ill-posed (the solution does not depend continuously on the data), if $R(T)$ is not closed. Ill-posed problems have to be solved by regularization methods, e.g., Tikhonov-regularization (cf. [4]). The idea of the Tikhonov-regularization is to approximate $T^\dagger y$ by the minimizing element of the functional

$$\phi_\alpha(x) := \|Tx - y\|^2 + \alpha \|x\|^2, \qquad \alpha > 0. \tag{2.5}$$

If $C(\subset X)$ is a closed convex set, we regularize the problem of solving (2.1) by solving the minimization problem

$$\min_{x \in C} \phi_\alpha(x), \qquad \alpha > 0, \tag{2.6}$$

where ϕ_α is defined by (2.5).

We show that the problem (2.6) has a unique solution for all $\alpha > 0$ and that these solutions converge to the C-best-approximate solution of (2.1) for $\alpha \to 0$ if $Ry \in T(C)$. Moreover, we show that for all $\alpha > 0$ the solution of problem (2.6) depends continuously on the data y. Therefore, the problem of solving (2.6) is well-posed. The existence of a unique solution of (2.6), the convergence of these solutions to the C-best-approximate solution of (2.1) for $\alpha \to 0$, and the stability of these solutions for fixed $\alpha > 0$ have been shown in [5] for the case $y \in T(C)$ and in [6] for the case $Ry \in T(C)$. Our proofs differ from those in [5, 6] and have been developed independently.

THEOREM 2.3. *Let $T \in L(X, Y)$, $y \in Y$, $\alpha > 0$, and $C(\subset X)$ be a convex closed set. Then the problem* (2.6) *has a unique solution $x_{\alpha,C}$. $x_{\alpha,C}$ is also the unique solution of problem* (2.6) *with y replaced by Qy, where Q is the orthogonal projector of Y onto $\overline{R(T)}$.*

Proof. It follows with (2.5) that ϕ_α is a strictly convex and Fréchet-differentiable functional with $\lim_{\|x\| \to \infty} \phi_\alpha(x) = \infty$ and $\nabla\phi_\alpha(x) = 2(T^*Tx + \alpha x - T^*y)$ for all $\alpha > 0$. Hence by the Kuhn–Tucker theory, problem (2.6) has a unique solution $x_{\alpha,C}$, which is characterized as the unique element in C, such that the variational inequality

$$(T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y, h - x_{\alpha,C}) \geqslant 0 \qquad \text{for all } h \in C \tag{2.7}$$

holds. Since $T^* = T^*Q$, it follows that $x_{\alpha,C}$ is also the minimizing element of ϕ_α with y replaced by Qy. ∎

In the next theorem we show that the solution $x_{\alpha,C}$ $(\alpha > 0)$ of the problems (2.6) converge to an element in C if and only if $Ry \in T(C)$ and that, if $Ry \in T(C)$, the limit point equals the C-best-approximate solution $x_{0,C}$ of (2.1). We call $x_{\alpha,C}$ "*constrained Tikhonov-regularized solution*" of (2.1).

THEOREM 2.4. *Let $T \in L(X, Y)$, $y \in Y$.*

(a) *The constrained Tikhonov-regularized solutions $x_{\alpha,C}$ converge to an element in C for $\alpha \to 0$ if and only if $Ry \in T(C)$.*

(b) *$Ry \in T(C)$ implies that $\lim_{\alpha \to 0} x_{\alpha,C} = x_{0,C}$.*

Proof. Let $\bar{x} \in C$ be such that $\lim_{\alpha \to 0} x_{\alpha,C} = \bar{x}$. Then it follows with (2.7) that $(T^*T\bar{x} - T^*y, h - \bar{x}) = (T\bar{x} - y, Th - T\bar{x}) \geqslant 0$ for all $h \in C$. Since $T(C)$ is dense in $\overline{T(C)}$, this implies that $(T\bar{x} - y, u - T\bar{x}) \geqslant 0$ for all $u \in \overline{T(C)}$. Now we obtain with (2.3) that $T\bar{x} = Ry$, which together with $\bar{x} \in C$ implies that $Ry \in T(C)$.

Now we assume that $Ry \in T(C)$. We show that $\lim_{\alpha \to 0} x_{\alpha,C} = x_{0,C}$. The existence of $x_{0,C}$ follows from Proposition 2.2. The definition of $x_{0,C}$ (cf. Definition 2.1) and (2.4) imply

$$Tx_{0,C} = Ry. \tag{2.8}$$

Now the definition of Ry and (2.8) imply that

$$\|Tx - y\| \geqslant \|Tx_{0,C} - y\| \qquad \text{for all } x \in C. \tag{2.9}$$

By definition of $x_{\alpha,C}$,

$$\begin{aligned} &\|Tx_{\alpha,C} - y\|^2 - \|Tx_{0,C} - y\|^2 + \alpha\, \|x_{\alpha,C}\|^2 \\ &\quad \leqslant \|Tx_{0,C} - y\|^2 - \|Tx_{0,C} - y\|^2 + \alpha\, \|x_{0,C}\|^2. \end{aligned}$$

Together with (2.9) this implies that

$$\|x_{\alpha,C}\| \leqslant \|x_{0,C}\| \qquad \text{for all } \alpha > 0. \tag{2.10}$$

With (2.7) (with $h = x_{0,C}$), (2.3) (with $u = Tx_{\alpha,C}$), and (2.8) we get

$$\frac{\left.\begin{aligned} (T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y,\, x_{0,C} - x_{\alpha,C}) \geqslant 0 \\ (T^*y - T^*Tx_{0,C},\, x_{0,C} - x_{\alpha,C}) \geqslant 0 \end{aligned}\right\} +}{(T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*Tx_{0,C},\, x_{0,C} - x_{\alpha,C}) \geqslant 0,}$$

which implies that

$$\|T(x_{\alpha,C}-x_{0,C})\|^2 \leqslant \alpha(x_{\alpha,C}, x_{0,C}-x_{\alpha,C}). \tag{2.11}$$

Inequalities (2.10) and (2.11) imply that $\|T(x_{\alpha,C}-x_{0,C})\| \leqslant (2\alpha)^{1/2} \|x_{0,C}\|$ and hence

$$\lim_{\alpha \to 0} Tx_{\alpha,C} = Tx_{0,C}. \tag{2.12}$$

Inequality (2.10) also implies that the weak closure of the set $\{x_{\alpha,C} \mid \alpha > 0\}$ is weakly compact and hence (cf. [1]) weakly sequentially compact. Let $(\alpha_n > 0)$ be an aribitrary sequence with $\alpha_n \to 0$ for $n \to \infty$. Then there exist a subsequence (again denoted by (α_n)) and an element u with $\|u\| \leqslant \|x_{0,C}\|$ and $x_{\alpha_n,C} \rightharpoonup u$, where "$\rightharpoonup$" denotes weak convergence. Since C is weakly closed (cf. [1]), $u \in C$. Equalities (2.12) and (2.8) imply that $Tu = Tx_{0,C} = Ry$. Since $x_{0,C}$ is the unique element of minimal norm among all elements $x \in C$ with $Tx = Ry$, it follows with $\|u\| \leqslant \|x_{0,C}\|$, $u \in C$, and $Tu = Ry$ that $u = x_{0,C}$. Therefore, we have shown that

$$x_{\alpha,C} \rightharpoonup x_{0,C} \quad \text{for} \quad \alpha \to 0. \tag{2.13}$$

With (2.13) and (2.10) we get

$$\begin{aligned}\|x_{0,C}\|^2 = \lim_{\alpha \to 0} |(x_{\alpha,C}, x_{0,C})| &\leqslant \liminf_{\alpha \to 0} \|x_{0,C}\| \cdot \|x_{\alpha,C}\| \\ &\leqslant \limsup_{\alpha \to 0} \|x_{0,C}\| \cdot \|x_{\alpha,C}\| \leqslant \|x_{0,C}\|^2\end{aligned}$$

and hence $\lim_{\alpha \to 0} \|x_{\alpha,C}\| = \|x_{0,C}\|$. Together with (2.13) this implies that

$$\lim_{\alpha \to 0} x_{\alpha,C} = x_{0,C}. \quad \blacksquare$$

In the next theorem we show that the constrained Tikhonov-regularized solution $x_{\alpha,C}$ depends Lipschitz-continuously on the data y.

THEOREM 2.5. *Let $\alpha > 0$, and let $x_{\alpha,C}$ and $\bar{x}_{\alpha,C}$ be the constrained Tikhonov-regularized solutions for the right-hand sides y and $\bar{y}$ of Eq.* (2.1), *respectively, and let Q be the orthogonal projector onto $R(T)$. Then*

$$\|x_{\alpha,C} - \bar{x}_{\alpha,C}\| \leqslant \frac{\|Q(y-\bar{y})\|}{\alpha^{1/2}} \quad \text{and} \quad \|T(x_{\alpha,C} - \bar{x}_{\alpha,C})\| \leqslant \|Q(y-\bar{y})\|$$

hold.

Proof. With (2.7) we get

$$\left.\begin{array}{l}(T^*Tx_{\alpha,C}+\alpha x_{\alpha,C}-T^*y,\ \bar{x}_{\alpha,C}-x_{\alpha,C})\geqslant 0\\ (T^*T\bar{x}_{\alpha,C}+\alpha\bar{x}_{\alpha,C}-T^*\bar{y},\ x_{\alpha,C}-\bar{x}_{\alpha,C})\geqslant 0\end{array}\right\}+$$

$$\overline{((T^*T+\alpha I)(x_{\alpha,C}-\bar{x}_{\alpha,C})+T^*(\bar{y}-y),\ \bar{x}_{\alpha,C}-x_{\alpha,C})\geqslant 0}$$

and hence with $T^*=T^*Q$

$$\|T(\bar{x}_{\alpha,C}-x_{\alpha,C})\|^2+\alpha\ \|\bar{x}_{\alpha,C}-x_{\alpha,C}\|^2\leqslant (Q(\bar{y}-y),\ T(\bar{x}_{\alpha,C}-x_{\alpha,C}))$$
$$\leqslant \|Q(\bar{y}-y)\|\cdot\|T(\bar{x}_{\alpha,C}-x_{\alpha,C})\|.$$

Our assertions follow from the last inequality. ▮

For some further properties of constrained Tikhonov-regularized solutions see [7, Theorem 1.7].

In the unconstrained case ($C=X$) it holds that, if there exists an $\bar{\alpha}>0$ such that, $x_{\bar{\alpha}}=x_0$ (where $x_\alpha=(T^*T+\alpha I)^{-1}T^*y$ is the unconstrained Tikhonov-regularized solution in X and $x_0=T^\dagger y$ is the best-approximate solution of (2.1) in X), then $x_\alpha=x_0=0$ for all $\alpha>0$. The next theorem shows that an analogous assertion holds for the constrained case.

THEOREM 2.6. *Let $Ry\in T(C)$, R and Q as above. By x_C we denote the unique element of minimal norm in C.*

(a) *$x_{0,C}=x_C$ implies that $x_{\alpha,C}=x_{0,C}=x_C$ for all $\alpha>0$.*

(b) *Let $Ry=Qy$. If there exists an $\bar{\alpha}>0$ such that $x_{\bar{\alpha},C}=x_{0,C}$, then $x_{\alpha,C}=x_{0,C}=x_C$ for all $\alpha>0$.*

(c) *Let $Ry\neq Qy$. If there exists an $\bar{\alpha}>0$ such that $x_{\bar{\alpha},C}=x_{0,C}$, then $x_{\alpha,C}=x_{0,C}$ for all $0<\alpha\leqslant\bar{\alpha}$.*

Proof. The existence and uniqueness of x_C follow immediately from the convexity of the closed set C.

(a) With (2.8) and (2.3) we obtain that $(T^*Tx_{0,C}-T^*y,\ h-x_{0,C})\geqslant 0$ for all $h\in C$. It follows from $x_{0,C}=x_C$ and the definition of x_C that $(x_{0,C},\ h-x_{0,C})\geqslant 0$ for all $h\in C$. These two inequalities imply that

$$(T^*Tx_{0,C}-T^*y,\ h-x_{0,C})+\alpha(x_{0,C},\ h-x_{0,C})$$
$$=(T^*Tx_{0,C}+\alpha x_{0,C}-T^*y,\ h-x_{0,C})\geqslant 0$$

for all $h\in C$ and $\alpha>0$. Hence the uniqueness of $x_{\alpha,C}$ and (2.7) imply that $x_{\alpha,C}=x_{0,C}$ for all $\alpha>0$.

(b) Let $\bar{\alpha} > 0$ be such that $x_{\bar{\alpha},C} = x_{0,C}$. Equality (2.8) and $Ry = Qy$ imply that $T^*Tx_{0,C} = T^*Ry = T^*Qy = T^*y$. Together with $x_{\bar{\alpha},C} = x_{0,C}$ and (2.7) we obtain that $\bar{\alpha}(x_{0,C}, h - x_{0,C}) \geqslant 0$ for all $h \in C$, which implies that $x_{0,C} = x_C$. The rest follows from (a).

(c) Let $\bar{\alpha} > 0$ be such that $x_{\bar{\alpha},C} = x_{0,C}$. Then (2.8) and (2.7) imply that $(Ry - y,\ Th - Ry) + \bar{\alpha}(x_{0,C}, h - x_{0,C}) = (T^*Tx_{0,C} + \bar{\alpha}x_{0,C} - T^*y, h - x_{0,C}) \geqslant 0$ for all $h \in C$. This inequality implies that, if $h \in C$ is such that $(x_{0,C}, h - x_{0,C}) < 0$, then $(Ry - y, Th - Ry) + \alpha(x_{0,C}, h - x_{0,C}) \geqslant (Ry - y, Th - Ry) + \bar{\alpha}(x_{0,C}, h - x_{0,C}) \geqslant 0$ for all $0 < \alpha \leqslant \bar{\alpha}$. If $h \in C$ is such that $(x_{0,C}, h - x_{0,C}) \geqslant 0$, then (2.3) implies that $(Ry - y, Th - Ry) + \alpha(x_{0,C}, h - x_{0,C}) \geqslant 0$ for all $\alpha > 0$. Together with (2.8) we obtain $(Ry - y, Th - Ry) + \alpha(x_{0,C}, h - x_{0,C}) = (T^*Tx_{0,C} + \alpha x_{0,C} - T^*y, h - x_{0,C}) \geqslant 0$ for all $h \in C$ and $0 < \alpha \leqslant \bar{\alpha}$. Now the uniqueness of $x_{\alpha,C}$ and (2.7) imply that $x_{\alpha,C} = x_{0,C}$ for all $0 < \alpha \leqslant \bar{\alpha}$. ∎

If instead of the exact right-hand side y in Eq. (2.1) we only know perturbed data $y_\delta \in Y$ such that $\|Q(y - y_\delta)\| \leqslant \delta$, then the following theorem shows how α has to be chosen in dependence on δ, that $x^\delta_{\alpha,C} \to x_{0,C}$ for $\delta \to 0$ holds, where $x^\delta_{\alpha,C}$ is the constrained Tikhonov-regularized solution of (2.1) with y replaced by y_δ.

THEOREM 2.7. *Let $Ry \in T(C)$ and $y_\delta \in Y$ such that $\|Q(y - y_\delta)\| \leqslant \delta$. If $\alpha(\delta)$ is such that $\lim_{\delta \to 0} \alpha(\delta) = 0$ and $\lim_{\delta \to 0} (\delta^2/\alpha(\delta)) = 0$, then $\lim_{\delta \to 0} x^\delta_{\alpha(\delta),C} = x_{0,C}$ holds.*

Proof. With Theorem 2.5 we get

$$\begin{aligned}\|x^\delta_{\alpha(\delta),C} - x_{0,C}\| &\leqslant \|x_{\alpha(\delta),C} - x_{0,C}\| + \|x^\delta_{\alpha(\delta),C} - x_{\alpha(\delta),C}\| \\ &\leqslant \|x_{\alpha(\delta),C} - x_{0,C}\| + \delta \cdot \alpha(\delta)^{-1/2}.\end{aligned}$$

The rest follows with Theorem 2.4. ∎

3. Convergence Rates for the Unconstrained Case

In this section we summarize some well-known convergence results for unconstrained Tikhonov-regularized solutions so that we can compare convergence results for constrained Tikhonov-regularized solutions, which we derive in the next two sections, with those of the unconstrained case. The proofs of the following assertions can be found, e.g., in [4, 9] (cf. also [2, 3]). We denote the Tikhonov-regularized solution in X by x_α and the best-approximate solution $T^\dagger y$ by x_0, if $y \in D(T^\dagger)$:

If $x_0 \in R(T^*)$, then $\|x_\alpha - x_0\| = o(\alpha^{1/2})$ and $\|T(x_\alpha - x_0)\| = O(\alpha)$.

If $x_0 \in R(T^*T)$, then $\|x_\alpha - x_0\| = O(\alpha)$.

Let $\nu \in (0, 1)$; if $x_0 \in R((T^*T)^\nu)$, then $\|x_\alpha - x_0\| = o(\alpha^\nu)$ and

$$\|T(x_\alpha - x_0)\| = \begin{cases} o(\alpha^{\nu+1/2}) & \text{if} \quad \nu < \frac{1}{2} \\ O(\alpha) & \text{if} \quad \nu \geqslant \frac{1}{2}. \end{cases}$$

Since $\|x_\alpha - x_0\| = o(\alpha)$ implies that $x_\alpha = x_0 = 0$ for all $\alpha > 0$, $O(\alpha)$ is (except for trivial cases) the best possible convergence rate.

Let x_α^δ be the Tikhonov-regularized solution in X with y replaced by perturbed data y_δ (such that $\|Q(y - y_\delta)\| \leqslant \delta$):

If $x_0 \in R(T^*)$ and $\alpha(\delta) \sim \delta$, then $\|x_{\alpha(\delta)}^\delta - x_0\| = O(\delta^{1/2})$.

Let $\nu \in (0, 1]$; if $x_0 \in R((T^*T)^\nu)$ and $\alpha(\delta) \sim \delta^{2/(2\nu+1)}$, then $\|x_{\alpha(\delta)}^\delta - x_0\| = O(\delta^{2\nu/(2\nu+1)})$.

The converse result in the following theorem is slightly more general than the converse result in [4].

THEOREM 3.1. *Let $y \in D(T^\dagger)$ and $\nu \in (0, 1]$. $x_0 \in R((T^*T)^\nu)$ if and only if $\|(T^*T)^{1-\nu}(x_\alpha - x_0)\| = O(\alpha)$; $x_0 \in R(T^*)$ if and only if $\|T(x_\alpha - x_0)\| = O(\alpha)$.*

Proof. This assertion is well known in the case where $\nu = 1$ and T is compact (see [4]). We now show for $\nu \in (0, 1]$:

$$x_0 \in R((T^*T)^\nu) \text{ if and only if } \|(T^*T)^{1-\nu}(x_\alpha - x_0)\| = O(\alpha).$$

"$x_0 \in R(T^*) \Leftrightarrow \|T(x_\alpha - x_0)\| = O(\alpha)$" is proven analogously.

"$\Rightarrow$": Let u be such that $(T^*T)^\nu u = x_0$ and let $\{E_\lambda \mid \lambda \geqslant 0\}$ be the spectral family of T^*T; then

$$\begin{aligned}(T^*T)^{1-\nu}(x_\alpha - x_0) &= (T^*T)^{1-\nu}[(T^*T + \alpha I)^{-1}\, T^*T - I](T^*T)^\nu u \\ &= -\alpha\left(\int_0^\infty \underbrace{\frac{\lambda}{\alpha + \lambda}}_{\leqslant 1} dE_\lambda\right) u,\end{aligned}$$

which implies that $\|(T^*T)^{1-\nu}(x_\alpha - x_0)\| \leqslant \alpha\|u\|$.

"$\Leftarrow$": It follows from $T^*Tx_\alpha + \alpha x_\alpha - T^*y = 0$ and $T^*y = T^*Tx_0$ for $\nu \leqslant 1$ that

$$x_\alpha = (T^*T)^\nu((1/\alpha)(T^*T)^{1-\nu}(x_0 - x_\alpha)). \tag{3.1}$$

Let $\nu \in (0, 1]$ be such that $\|(T^*T)^{1-\nu}(x_\alpha - x_0)\| = O(\alpha)$. Thus, there exists a constant $\gamma \geqslant 0$ such that

$$\|(1/\alpha)(T^*T)^{1-\nu}(x_0 - x_\alpha)\| \leqslant \gamma. \tag{3.2}$$

Let $(\alpha_n > 0)$ be an arbitrary sequence such that $\alpha_n \to 0$ for $n \to \infty$; since the set $\{x \in X / \|x\| \leqslant \gamma\}$ is weakly sequentially compact (cf. [1]), (3.2) implies that there exist a subsequence (again denoted by (α_n)) and an element $f \in X$ such that $(1/\alpha_n)(T^*T)^{1-\nu}(x_0 - x_{\alpha_n}) \rightharpoonup f$ for $n \to \infty$ and hence

$$(T^*T)^\nu((1/\alpha_n)(T^*T)^{1-\nu}(x_0 - x_{\alpha_n})) \rightharpoonup (T^*T)^\nu f \quad \text{for} \quad n \to \infty. \tag{3.3}$$

With (3.1), (3.3), and $x_\alpha \to x_0$ for $\alpha \to 0$, we obtain $x_0 = (T^*T)^\nu f$, i.e., $x_0 \in R((T^*T)^\nu)$. ▮

4. Conditions for the Rate $o(\alpha^{1/2})$ in the Constrained Case

In this section we show that the condition "$x_0 \in R(T^*)$" that yields the convergence rate $o(\alpha^{1/2})$ in the unconstrained case (see Section 3) can be replaced by the condition "$x_{0,C} \in R(P_C T^*)$" in the constrained case, where P_C is the metric projector onto C, to yield the same result.

Let $Ry \in T(C)$; we define

$$N := \{x \in N(T)^\perp / P_C x = x_{0,C}\} \tag{4.1}$$

and

$$U := \{u \in \overline{R(T)} / P_C T^* u = x_{0,C}\}. \tag{4.2}$$

Lemma 4.1. *Let N and U be as above. Then N and U are closed and convex. If $U \neq \emptyset$, then there exists a unique element $\bar{u}$ of minimal norm in U. Moreover, $\overline{T^*U} \subset N$.*

Proof. Let $N \neq \emptyset$, which is equivalent to $x_{0,C} \in P_C(N(T)^\perp)$, and let (x_n) be an arbitrary sequence in N such that $x_n \to x \in N(T)^\perp$ for $n \to \infty$. Since

$$x \in N \Leftrightarrow x \in N(T)^\perp \text{ and } (x_{0,C} - x, h - x_{0,C}) \geqslant 0 \text{ for all } h \in C \tag{4.3}$$

holds, we obtain $0 \leqslant (x_{0,C} - x_n, h - x_{0,C}) \to_{n \to \infty} (x_{0,C} - x, h - x_{0,C})$ for all $h \in C$, and hence $(x_{0,C} - x, h - x_{0,C}) \geqslant 0$ for all $h \in C$. Again (4.3) implies that $x \in N$. Therefore, N is closed.

Let now $x_1, x_2 \in N$ and $\lambda \in [0, 1]$; it follows with (4.3) that

$$\begin{aligned}&(x_{0,C} - (\lambda x_1 + (1-\lambda) x_2), h - x_{0,C}) \\ &\quad = \underbrace{\lambda}_{\geqslant 0} \underbrace{(x_{0,C} - x_1, h - x_{0,C})}_{\geqslant 0} + \underbrace{(1-\lambda)}_{\geqslant 0} \underbrace{(x_{0,C} - x_2, h - x_{0,C})}_{\geqslant 0} \geqslant 0\end{aligned}$$

for all $h \in C$, and hence with (4.3) that $(\lambda x_1 + (1-\lambda) x_2) \in N$. Therefore, N is convex. One proves analogously that U is closed and convex, using

$$u \in U \Leftrightarrow u \in \overline{R(T)} \text{ and } (x_{0,C} - T^*u, h - x_{0,C}) \geqslant 0 \text{ for all } h \in C \qquad (4.4)$$

instead of (4.3). If $U \neq \varnothing$, the existence and uniqueness of an element $\bar{u}$ of minimal norm in U follow from the convexity of the closed set U. $\overline{T^*U} \subset N$ follows immediately from the definitions of N and U, since $R(T^*) \subset N(T)^\perp$. ▮

Now we can prove our first result about convergence rates:

THEOREM 4.2. *Let $Ry \in T(C)$.*

(a) *If $x_{0,C} \in R(P_C T^*)$, then $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha^{1/2})$ and $\|T(x_{\alpha,C} - x_{0,C})\| = O(\alpha)$. If in addition $Qy = Ry$ we even obtain $\|x_{\alpha,C} - x_{0,C}\| = o(\alpha^{1/2})$.*

(b) *Let $Qy = Ry$; $\|T(x_{\alpha,C} - x_{0,C})\| = O(\alpha)$ implies that $x_{0,C} \in R(P_C T^*)$.*

Proof. (a) It follows with (2.7) (with $h = x_{0,C}$), (2.3) (with $u = Tx_{\alpha,C}$), (2.8), and (4.4) (with $h = x_{\alpha,C}$) that

$$\frac{\left.\begin{array}{r} (T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y, x_{0,C} - x_{\alpha,C}) \geqslant 0 \\ (T^*y - T^*Tx_{0,C}, x_{0,C} - x_{\alpha,C})) \geqslant 0 \\ \alpha(T^*\bar{u} - x_{0,C}, x_{0,C} - x_{\alpha,C}) \geqslant 0 \end{array}\right\} +}{((T^*T + \alpha I)(x_{\alpha,C} - x_{0,C}) + \alpha T^*\bar{u}, x_{0,C} - x_{\alpha,C}) \geqslant 0,}$$

where $\bar{u} \in U$ is the unique element of minimal norm in U (see Lemma 4.1).

The last inequality implies that

$$\|T(x_{\alpha,C} - x_{0,C})\|^2 + \alpha\, \|x_{\alpha,C} - x_{0,C}\|^2 \leqslant \alpha(\bar{u}, T(x_{0,C} - x_{\alpha,C})) \leqslant \alpha\, \|\bar{u}\| \cdot \|T(x_{\alpha,C} - x_{0,C})\|. \qquad (4.5)$$

It follows immediately from (4.5) that

$$\|T(x_{\alpha,C} - x_{0,C})\| \leqslant \alpha\, \|\bar{u}\| \qquad (4.6)$$

and

$$\|x_{\alpha,C} - x_{0,C}\| \leqslant \alpha^{1/2}\, \|\bar{u}\| \qquad (4.7)$$

hold. Let now $Qy = Ry$. Estimate (4.6) implies that $\|T(x_{\alpha,C} - x_{0,C})/\alpha\| \leqslant \|\bar{u}\|$. Let $(\alpha_n > 0)$ be an arbitrary sequence with $\alpha_n \to 0$ for $n \to \infty$. Then there exist a subsequence (again denoted by (α_n)) and an element $g_0 \in \overline{R(T)}$ such that $\|g_0\| \leqslant \|\bar{u}\|$ and $T(x_{0,C} - x_{\alpha_n,C})/\alpha_n \rightharpoonup g_0$ for $n \to \infty$. Together with (2.7), $Ry = Qy$, and (2.8), which imply that $T^*y = T^*Ry = T^*Tx_{0,C}$, we obtain

$$0 \leqslant \left(x_{\alpha_n,C} - T^* \frac{T(x_{0,C} - x_{\alpha_n,C})}{\alpha_n}, h - x_{\alpha_n,C} \right) \xrightarrow[n \to \infty]{} (x_{0,C} - T^*g_0, h - x_{0,C})$$

for all $h \in C$ and hence $(x_{0,C} - T^*g_0, h - x_{0,C}) \geqslant 0$ for all $h \in C$. Now (4.4) implies that $g_0 \in U$. $\|g_0\| \leqslant \|\bar{u}\|$ and the uniqueness of $\bar{u}$ imply that $g_0 = \bar{u}$. Therefore, we have shown

$$\frac{T(x_{0,C} - x_{\alpha,C})}{\alpha} \rightharpoonup \bar{u} \qquad \text{for} \quad \alpha \to 0. \tag{4.8}$$

With (4.6) and (4.8) we get

$$\begin{aligned} \|\bar{u}\|^2 &= \lim_{\alpha \to 0} \left| \left(\bar{u}, \frac{T(x_{0,C} - x_{\alpha,C})}{\alpha} \right) \right| \\ &\leqslant \|\bar{u}\| \cdot \liminf_{\alpha \to 0} \left\| \frac{T(x_{0,C} - x_{\alpha,C})}{\alpha} \right\| \\ &\leqslant \|\bar{u}\| \cdot \limsup_{\alpha \to 0} \left\| \frac{T(x_{0,C} - x_{\alpha,C})}{\alpha} \right\| \leqslant \|\bar{u}\|^2 \end{aligned}$$

and hence $\lim_{\alpha \to 0} \|T(x_{0,C} - x_{\alpha,C})/\alpha\| = \|\bar{u}\|$. Together with (4.8) this implies

$$\frac{T(x_{0,C} - x_{\alpha,C})}{\alpha} \to \bar{u} \qquad \text{for} \quad \alpha \to 0. \tag{4.9}$$

Rewriting (4.5) we get

$$\alpha \|x_{\alpha,C} - x_{0,C}\|^2 \leqslant \alpha(\bar{u}, T(x_{0,C} - x_{\alpha,C})) - \|T(x_{0,C} - x_{\alpha,C})\|^2$$

and hence

$$\begin{aligned} \|x_{\alpha,C} - x_{0,C}\|^2 &\leqslant \left(\bar{u} - \frac{T(x_{0,C} - x_{\alpha,C})}{\alpha}, T(x_{0,C} - x_{\alpha,C}) \right) \\ &\leqslant \left\| \bar{u} - \frac{T(x_{0,C} - x_{\alpha,C})}{\alpha} \right\| \cdot \|T(x_{0,C} - x_{\alpha,C})\|. \end{aligned}$$

With (4.6) and (4.9) this implies that $\|x_{\alpha,C} - x_{0,C}\| = o(\alpha^{1/2})$.

(b) If $\|T(x_{\alpha,C} - x_{0,C})\| = O(\alpha)$, then there exists a constant $\gamma > 0$ such that $\|T(x_{\alpha,C} - x_{0,C})/\alpha\| \leqslant \gamma$. Now it follows analogously to (a) that an arbitrary sequence ($\alpha_n > 0$) with $\alpha_n \to 0$ for $n \to \infty$ has a subsequence (again denoted by (α_n)) such that $T(x_{0,C} - x_{\alpha_n,C})/\alpha_n \rightharpoonup g_0 \in U$ for $n \to \infty$, i.e., $P_C T^* g_0 = x_{0,C}$, so that $x_{0,C} \in R(P_C T^*)$. ▮

Remark 4.3. If the operator T is injective and $Qy = Ry$, which implies that $x_{0,C} = x_0 = T^\dagger y$, then the condition "$x_{0,C} = x_0 \in R(P_C T^*)$" for the convergence rate $o(\alpha^{1/2})$ in the constrained case is weaker than the condition "$x_0 \in R(T^*)$" in the unconstrained case, since there exist examples (see, e.g., [7, Example 3.4]) where $x_0 \notin R(T^*)$, $\|x_\alpha - x_0\| \neq O(\alpha^{1/2})$, but $x_{0,C} = x_0 \in R(P_C T^*)$, which implies (see Theorem 4.2) that $\|x_{\alpha,C} - x_0\| = o(\alpha^{1/2})$. This means that in this case the constrained Tikhonov-regularized solution converges faster than the unconstrained Tikhonov-regularized solution. Obviously, the converse implication "$x_0 \in R(T^*) \Rightarrow x_0 \in R(P_C T^*)$" always holds, if $x_0 \in C$.

It is also possible to show that the condition "$x_0 \in R((T^*T)^\nu)$ $(\nu < \frac{1}{2})$" for the convergence rate $o(\alpha^\nu)$ in the unconstrained case (see Section 3) can be replaced by an analogous condition for $x_{0,C}$ in the constrained case (see [7, pp. 32–39]).

5. Conditions for the Rate $O(\alpha)$ in the Constrained Case

In this section we summarize the most important results of [7]. Since the proofs of these results are rather involved, they will be omitted here; for the proofs see [7].

The first result in this section shows that we can use the results about convergence rates of the unconstrained case, if $x_{0,C} \in \mathring{C}$ (which is not surprising):

Proposition 5.1. *Let* $Ry \in T(C)$ *and* $x_{0,C} \in \mathring{C}$. *Then* $x_{0,C} = x_0$ *and* $x_{\alpha,C} = x_\alpha$ *for* $\alpha > 0$ *sufficiently small.*

Proof. See [7, Proposition 4.1]. ▮

In the following we suppose that $x_{0,C} \in \partial C$, which holds, e.g., if $x_{0,C} \neq x_0$ or $Qy \neq Ry$.

In the unconstrained case, $O(\alpha)$ is (except for trivial cases, cf. Section 3) the best possible convergence rate. This also holds in the constrained case (at least if $Ry = Qy$):

Theorem 5.2. *Let* $Ry = Qy \in T(C)$ *and* $\|x_{\alpha,C} - x_{0,C}\| = o(\alpha)$; *then* $x_{\alpha,C} = x_{0,C} = x_C$ *for all* $\alpha > 0$, *where* x_C *is the unique element of minimal norm in C.*

Proof. See [7, Theorem 4.2]. ▮

We have seen in Section 4 that the condition "$x_0 \in R(T^*)$" for the convergence rate $o(\alpha^{1/2})$ in the unconstrained case can be replaced by the condition "$x_{0,C} \in R(P_C T^*)$" in the constrained case to yield the same result. The condition "$x_0 \in R(T^*T)$" for the convergence rate $O(\alpha)$ in the unconstrained case cannot just be replaced by "$x_{0,C} \in R(P_C T^*T)$" as one would expect (for an example where $x_{0,C} \in R(P_C T^*T)$, but $\|x_{\alpha,C} - x_{0,C}\| \neq O(\alpha)$, see [7, Example 4.10]). Nevertheless, the next theorem shows that this condition is necessary for the convergence rate $O(\alpha)$, if $Qy = Ry$.

THEOREM 5.3. *Let $Qy = Ry \in T(C)$, $x_{0,C} \in \partial C$, $x_{0,C} \neq x_C$, and $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha)$. Then $U \neq \emptyset$ and $\bar{u} \in R(T\tilde{P})$ holds, where U is defined by (4.2) and $\tilde{P}$ is the orthogonal projector onto $\tilde{L} := \{h \in X/(f_0, h)\} = 0$, where*

$$f_0 = \begin{cases} \dfrac{x_{0,C} - T^*\bar{u}}{\|x_{0,C} - T^*\bar{u}\|} & \text{if } x_{0,C} \neq T^*\bar{u} \\ 0 & \text{if } x_{0,C} = T^*\bar{u}. \end{cases}$$

Hence $x_{0,C} \in R(P_C T^*T\tilde{P})$. Especially, $x_{0,C} \in R(P_C T^*T)$.

Proof. See [7, Theorem 4.5]. ∎

We have seen in Remark 4.3 that, if T is injective and $x_0 \in C$, the constrained Tikhonov-regularized solutions always converge with the rate $o(\alpha^{1/2})$, if the unconstrained Tikhonov regularized solutions converge with this rate. An analogous assertion does not hold for the rate $O(\alpha)$, even if T is injective and $x_0 \in C$, because it is on the one hand possible that only the unconstrained Tikhonov-regularized solutions converge with the rate $O(\alpha)$, and on the other hand it is possible that only the constrained Tikhonov-regularized solutions converge with the rate $O(\alpha)$ (see [7, Example 4.7]).

If ∂C can be described by a linear manifold in a neighbourhood of $x_{0,C}$, we can use the convergence results from the "unconstrained theory" (see Section 3) to obtain the following result:

THEOREM 5.4. *Let $Ry \in T(C)$ and $x_{0,C} \in \partial C$. Let V be a linear subspace of X, $z \in V^\perp$, and $\varepsilon > 0$ such that $(z + V) \cap U_\varepsilon(x_{0,C}) = \partial C \cap U_\varepsilon(x_{0,C})$, where $U_\varepsilon(x_{0,C}) := \{x \in X/\|x - x_{0,C}\| < \varepsilon\}$. For $\alpha > 0$ sufficiently small let $x_{\alpha,C} \in \partial C$. Then*

$$Px_{\alpha,C} = (PT^*TP + \alpha I)^{-1} PT^*TPx_{0,C} \quad \text{and} \quad P(x_{\alpha,C} - x_{0,C}) = x_{\alpha,C} - x_{0,C}$$

for $\alpha > 0$ sufficiently small, where P is the orthogonal projector onto V. Moreover, $Px_{0,C} \in R(PT^*TP)$ is equivalent to $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha)$.

Proof. See [7, Theorem 4.8]. ∎

Theorem 5.4 shows that, if L is a hyperplane and $x_{\alpha,L}$ are the constrained regularized solutions in this hyperplane, we can use the results of Section 3 to obtain results about convergence rates. Therefore, we now establish conditions under which the inequality

$$\|x_{\alpha,C}-x_{0,C}\| \leqslant c\cdot\|x_{\alpha,L}-x_{0,C}\| \tag{5.1}$$

holds for some $c>0$, where $L:=\{h\in X/(T^*(Ry-Qy), h-x_{0,C})=0\}$, if $Qy\neq Ry$, and $L:=\{h\in X/(x_{0,C}-T^*\bar{u}, h-x_{0,C})=0\}$, if $Qy=Ry$, $x_{0,C}\in R(P_C T^*)$, and $x_{0,C}\neq T^*\bar{u}$ (see [7, Lemma 4.16]). The main effort in the proofs of the results in this section is contained in the derivation of sufficient conditions for (5.1) to hold. Using (5.1) and Theorem 5.4 one can obtain the convergence rate $O(\alpha)$ for a larger class of convex sets C. For the (lengthy) details see [7, Lemmata 4.12–4.16].

For $x\in\partial C$ let $S(x):=\{f\in X/(f, h-x)\geqslant 0$ for all $h\in C$ and $\|f\|=1\}$ be the subgradient of ∂C in x; and let

$$f_0:=\begin{cases}\dfrac{T^*(Ry-Qy)}{\|T^*(Ry-Qy)\|} & \text{if}\quad Ry\neq Qy\\[2ex] \dfrac{x_{0,C}-T^*\bar{u}}{\|x_{0,C}-T^*\bar{u}\|} & \text{if}\quad Ry=Qy,\quad x_{0,C}\in R(P_C T^*),\ \text{and}\ x_{0,C}\neq T^*\bar{u};\end{cases}$$

and $x_{\alpha,C}\neq x_{0,C}$ for all $\alpha>0$. Moreover, let either $Ry\neq Qy$ or $Ry=Qy$, $x_{0,C}\in R(P_C T^*)$ and $x_{0,C}\neq T^*\bar{u}$. Then the sufficient conditions for (5.1) read as follows:

"There exist constants $c>0$ and $\varepsilon>0$ such that

$$\frac{|(f_n-f_0, x_{0,C}-x_n)|}{\|f_n-f_0\|\cdot\|x_{0,C}-x_n\|}\geqslant c \qquad \text{for all sequences } (x_n), (f_n), \tag{5.2}$$

with $x_n \to_{n\to\infty} x_{0,C}$, $x_{0,C}\neq x_n\in\partial C\cap U_\varepsilon(x_{0,C})$, and $f_n\to_{n\to\infty} f_0, f_0\neq f_n\in S(x_n)$"

and

"There exist constants $c>0$ and $\varepsilon>0$ such that

$$\frac{|(f_n, x_{0,C}-x_n)|}{\|f_n-f_0\|\cdot\|x_{0,C}-x_n\|}\geqslant c \qquad \text{for all sequences } (x_n), (f_n), \tag{5.3}$$

with $x_n \to_{n\to\infty} x_{0,C}$, $x_{0,C}\neq x_n\in\partial C\cap U_\varepsilon(x_{0,C})$, and $f_n\to_{n\to\infty} f_0, f_0\neq f_n\in S(x_n)$,"

respectively. Since $(f_n, x_{0,C}-x_n)\geqslant 0$ and $(f_0, x_n-x_{0,C})\geqslant 0$ (note that $f_n\in S(x_n)$ and $f_0\in S(x_{0,C})$), $(f_n-f_0, x_{0,C}-x_n)=(f_n, x_{0,C}-x_n)+$

$(f_0, x_n - x_{0,C}) \geqslant (f_n, x_{0,C} - x_n) \geqslant 0$; hence $|(f_n - f_0, x_{0,C} - x_n)| = (f_n - f_0, x_{0,C} - x_n) \geqslant (f_n, x_{0,C} - x_n) = |(f_n, x_{0,C} - x_n)|$. This implies that

$$(5.3) \Rightarrow (5.2).$$

If C is twice continuously Fréchet-differentiable in a neighborhood of $x_{0,C}$, then there exists a very simple condition which implies (5.3).

LEMMA 5.5. *Let $Ry \in T(C)$ and let ∂C be twice continuously Fréchet-differentiable in a neighbourhood of $x_{0,C}$, i.e., there exist $\varepsilon > 0$, $c > 0$, and a functional $F: U_\varepsilon(x_{0,C}) \to \mathbb{R}$ such that $\partial C \cap U_\varepsilon(x_{0,C}) = \{x \in U_\varepsilon(x_{0,C}) / F(x) = c\}$ and F is twice continuously Fréchet-differentiable. If $F''(x_{0,C})$ is positive definite, then the conditions (5.2) and (5.3) hold. (By $F''(x_{0,C})$ is positive definite we mean that there exists a constant $\gamma > 0$ such that $F''(x_{0,C})(z, z) \geqslant \gamma \|z\|^2$ for all $z \in X$.)*

Proof. See [7, Lemma 4.18]. ▮

Now we obtain the main result of this section.

THEOREM 5.6. *Let $Ry \in T(C)$ and let one of the following four conditions be fulfilled.*

(i) $Ry \neq Qy$, $x_{0,C} \in N(T)^\perp$, *and* $Qy \in R(T)$
(ii) $Ry = Qy$, $x_{0,C} \neq x_0$, *and* $x_{0,C} \in R(P_C T^*)$
(iii) $Ry = Qy$, $x_{0,C} = x_0 \in R(P_C T^*)$, $x_{0,C} \neq T^*\bar{u}$, *and* $f_0 \in R(T^*T)$

} *and C fulfills condition* (5.2)

(iv) $Ry = Qy$, $x_{0,C} = x_0 \in R(P_C T^*)$, $x_{0,C} \neq T^*\bar{u}$, *and $f_0 \notin R(T^*T)$ and C fulfills condition* (5.3).

(If the conditions of Lemma 5.5 are fulfilled, (5.2) *and* (5.3) *hold.) Let $\tilde{P}$ be the orthogonal projector onto $\tilde{L} := \{h \in X / (f_0, h) = 0\}$, where*

$$f_0 = \begin{cases} \dfrac{T^*(Ry - Qy)}{\|T^*(Ry - Qy)\|} & \textit{in the case (i)} \\[2ex] \dfrac{x_{0,C} - T^*\bar{u}}{\|x_{0,C} - T^*\bar{u}\|} & \textit{in the cases (ii), (iii), (iv).} \end{cases}$$

Then $\tilde{P} x_{0,C} \in R(\tilde{P} T^* T \tilde{P})$ implies that

$$\|x_{\alpha,C} - x_{0,C}\| = O(\alpha).$$

Proof. See [7, Theorem 4.19].

If we do not know the data y exactly, but elements $y_\delta \in Y$ such that $\|Q(y - y_\delta)\| \leqslant \delta$, then we can obtain results about convergence rates in dependence on δ analogously to the unconstrained case (see Section 3), using Theorem 2.5, Theorem 4.2, Theorem 5.4, and Theorem 5.6 (see [7, Theorem 4.20]).

Note that in this paper we treat the infinite-dimensional theory of constrained regularization. For numerial computations one has to approximate the problem of solving (2.6) by a sequence of finite-dimensional problems. For this and numerical results see [8].

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